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Extremal graphs in some coloring problems

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Abstract

For a simple graph G with chromatic number $\chi(G)$, the Nordhaus–Gaddum inequalities give upper and lower bounds for $\chi(G)\chi(G^c)$ and $\chi(G)+\chi(G^c)$. Based on a characterization by Fink of the extremal graphs G attaining the lower bounds for the product and sum, we characterize the extremal graphs G for which $A(G)B(G^c)$ is minimum, where A and B are each of chromatic number, achromatic number and pseudoachromatic number. Characterizations are also provided for several cases in which $A(G)+B(G^c)$ is minimum. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

Throughout this paper, we consider only finite simple graphs. The notation and terminology are as in [1]. We recall some of these below: P_v is the path of length $v-1$; C_v the cycle of length v; K_v the complete graph on v vertices; $K(n_1,\dots,n_m)$ the complete m-partite graph with n_i vertices in the ith part, $1\leqslant i\leqslant m$; $K_m(n)$ the complete m-partite graph with n vertices in each part; $G\vee H$ the join of G and H; W_v the join of C_{v-1} and K_1; G^c the complement of G; $\chi(G)$ the chromatic number of G and $\omega(G)$ the number of components of G.

An achromatic k-coloring of a graph G is a proper vertex k-coloring of G in which every pair of distinct colors is represented by some edge. The maximum k for which G has an achromatic k-coloring is the achromatic number $\alpha(G)$ of G. A pseudoachromatic k-coloring of G is a k-coloring of G in which every pair of distinct colors is represented by some edge. The maximum k for which G has a pseudoachromatic k-coloring is the pseudoachromatic number $\psi(G)$ of G. It is clear that for any graph G, $\chi(G)\leqslant\alpha(G)\leqslant\psi(G)$. The celebrated Nordhaus–Gaddum [5] inequalities are:

$$v\leqslant\chi(G)\chi(G^c)\leqslant\lfloor((v+1)/2)^2\rfloor$$

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and

$$\lceil 2\sqrt{v}\rceil \leqslant \chi(G)+\chi(G^c)\leqslant v+1,$$

where $v=|V(G)|$.

Also, for the sum, the following upper bounds are known [2]:

$$\chi(G)+\psi(G^c)\leqslant v+1 \quad \text{and} \quad \psi(G)+\psi(G^c)\leqslant \lceil 4v/3\rceil .$$

Fink [3] characterized graphs G for which $\chi(G)\chi(G^c)=v$ and those for which $\chi(G)+\chi(G^c)=\lceil 2\sqrt{v}\rceil$. His solutions of these graph equations are based on the consideration of the set $T_1(v;x,y)$ of graphs with v vertices, where $x+y-1\leqslant v\leqslant xy$. The set $T_1(v;x,y)$ is defined as follows: Consider a rectangular array T with x rows and y columns and place at most one dot in each of the xy entries of T according to the following scheme: Place a dot in each entry of the first row and first column of T. This accounts for $x+y-1$ dots. Also place $v-(x+y-1)$ dots in any of the remaining entries of T. Then a graph G in $T_1(v;x,y)$ with v vertices is formed by taking the v dots of T as the vertices of G and defining adjacency in G as follows: (i) Any two dots in the same column are adjacent, (ii) no two dots in the same row are adjacent, and (iii) any two dots which belong to distinct rows and columns of G may or may not be adjacent. The two results of Fink are the following:

(i) $\chi(G)\chi(G^c)=v$ iff $G\in T_1(v;v/d,d)$, where d is a positive divisor of v.

(ii) $\chi(G)+\chi(G^c)=\lceil 2\sqrt{v}\rceil$ iff $G\in T_1(v;x,y)$ where $x+y=\lceil 2\sqrt{v}\rceil$.

It is clear that for any graph $G\in T_1(v;x,y)$, $\chi(G)=x$, $\chi(G^c)=y$ and that $G^c\in T_1(v;y,x)$.

In this paper, we determine completely the extremal graphs G for which $A(G)B(G^c)=v$, where $A,B\in\{\chi,\alpha,\psi\}$ and the extremal graphs G for which $\chi(G)+\psi(G^c)=\lceil 2\sqrt{v}\rceil$. As a consequence, we have improved the lower bound $\chi(G)+\psi(G^c)\geqslant\lceil 2\sqrt{v}\rceil$ by 1 for graphs G with at least 11 vertices. As v is a positive integer, there exists a positive integer m such that $m^2\leqslant v\leqslant m^2+2m$. Therefore we have:

$$\lceil 2\sqrt{v}\rceil = \begin{cases} 2m & \text{if } v=m^2,\\ 2m+1 & \text{if } m^2+1\leqslant v\leqslant m^2+m,\\ 2m+2 & \text{if } m^2+m+1\leqslant v\leqslant m^2+2m.\end{cases}$$

For $v\in\{m^2, m^2+m-1, m^2+m, m^2+2m\}$, we have completely determined the solution set for the graph equation $\chi(G)+\alpha(G^c)=\lceil 2\sqrt{v}\rceil$. For the other values of v, the determination of the extremal graphs for the above graph equation appears to be difficult.

2. Some preliminary results

Let $G\in T_1(v;x,y)$ and let there be a dot in the entry (i,j) of T_1. Then we denote the corresponding vertex by u_{ij} when regarded as a vertex of G and by v_{ji} when regarded as a vertex of G^c.

Lemma 1. *Let $x \geqslant 3$ and $y \geqslant 2$ be any two positive integers and let $G \in T_1(xy; x, y)$. Then $\alpha(G^c) = y$ iff $G \cong yK_x$.*

Proof. Clearly, G contains yK_x. Suppose that $\alpha(G^c) = y$ and $G \ncong yK_x$. Then in G^c, there exist two vertices, say, v_{ij} and v_{kl}, $i \neq k$, $j \neq l$, such that $(v_{ij}, v_{kl}) \notin E(G^c)$. Now color the vertices of G^c as follows: Color both the vertices v_{ij} and v_{kl} by c_{y+1} and for the remaining vertices color, v_{mn} by c_m. As $x \geqslant 3$, this coloring yields an achromatic $(y+1)$-coloring for G^c, contradicting the fact that $\alpha(G^c) = y$.

Conversely, if $G \cong yK_x$, then $G^c \cong K_y(x)$ and therefore $\alpha(G^c) = y$. □

Next we consider the case when $x = 2$.

Lemma 2. *Let $y \geqslant 2$ be any positive integer and let $G \in T_1(2y; 2, y)$. Then $\alpha(G^c) = y$ iff G is a disjoint union of complete regular bipartite graphs.*

Proof. Assume that $\alpha(G^c) = y$. We claim that for any four vertices u_{1k}, u_{2k}, u_{1l} and u_{2l}, the subgraph induced by them in G is isomorphic either to $2K_2$ or to C_4. Otherwise, the subgraph induced by them in both G as well as G^c is P_4. Suppose it is $u_{1l}u_{2l}u_{1k}u_{2k}$ in G. For $i = 1, 2$ and $j = 1, 2, \ldots, y$, color v_{ji} by c_j except the vertices v_{k1} and v_{l2}, to which we assign the color c_{y+1}. This yields an achromatic $(y+1)$-coloring for G^c, a contradiction.

We now obtain a new graph H from G as follows: Let $V(H) = \{u_1, u_2, \ldots, u_y\}$ with u_i joined to u_j in H iff the subgraph induced by $\{u_{1i}, u_{2i}, u_{1j}, u_{2j}\}$ in G is isomorphic to C_4. Now to show that each component of G is isomorphic to a complete regular bipartite graph, it is enough to show that H is a disjoint union of complete graphs. If this is not so, there must exist three vertices u_m, u_n and u_s such that $u_m u_s$, $u_n u_s \in E(H)$ and $u_m u_n \notin E(H)$. Now color the vertices of G^c as follows: Color v_{ji} by c_j, where $(j, i) \neq (m, 2), (s, 2), (n, 2)$ and color v_{m2}, v_{s2} and v_{n2} by c_{y+1}, c_m and c_s respectively. This yields an achromatic $(y+1)$-coloring for G^c, a contradiction.

Conversely, assume that $G \cong \bigcup_{i=1}^{l} K_{r_i, r_i}$. Since $\alpha(H_1 \vee H_2) = \alpha(H_1) + \alpha(H_2)$, we have $\alpha(G^c) = \alpha(2K_{r_1} \vee \cdots \vee 2K_{r_l}) = \sum_{i=1}^{l} \alpha(2K_{r_i}) = \sum_{i=1}^{l} r_i = y$. □

Next, we consider graphs $G \in T_1(xy - 1; x, y)$. As there exists exactly one gap in $T_1(xy - 1; x, y)$, without loss of generality, we can assume that the lone gap occurs at the position (x, y) of $T_1(xy - 1, x, y)$.

Lemma 3. *Let $x \geqslant 4$, $y \geqslant 2$ be any two positive integers and let $G \in T_1(xy - 1; x, y)$. Then $\alpha(G^c) = y$ iff $G \cong (y-1)K_x \cup K_{x-1}$ or $rK_x \cup H(x, y, r)$, with $y \geqslant r + 2$, where $H(x, y, r)$ is the graph of Fig. 1. ($K_{x-1} \vee \{w_i\}$ is displayed by $K_{x-1} = \{w_i\}$.)*

Proof. Suppose that $\alpha(G^c) = y$. By definition of T_1, G contains $(y-1)K_x \cup K_{x-1}$ as a subgraph. If $G \ncong (y-1)K_x \cup K_{x-1}$, then there exist two vertices, say, v_{ij} and v_{kl}, $i \neq k$, $j \neq l$, such that $(v_{ij}, v_{kl}) \notin E(G^c)$.

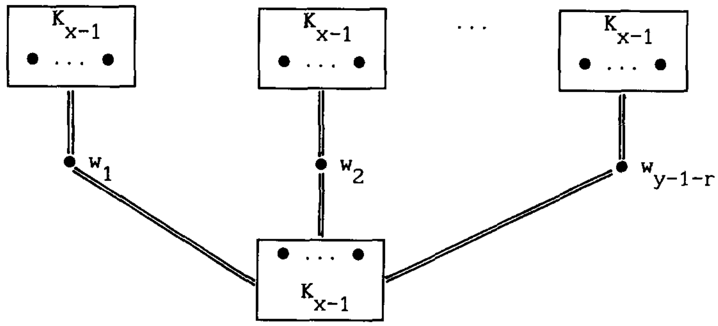

Fig. 1. $H(x,y,r)$

Claim. *Both the pairs (i,j) and (k,l) of vertices v_{ij} and v_{kl} are in the set $S=\{(1,x),(2,x),\ldots,(y-1,x),(y,1),(y,2),\ldots,(y,x-1)\}$.*

Suppose not. Then at least one of (i,j) and (k,l) is not in S, say, (i,j). Now color the vertices of G^c as follows: Color both the vertices v_{ij} and v_{kl} by c_{y+1} and the remaining vertices v_{mn} by c_m. As $x\geqslant 4$, this coloring yields an achromatic $(y+1)$-coloring for G^c, a contradiction.

Hence, by the above claim $(i,j)=(i,x)$ and $(k,l)=(y,l)$ or $(i,j)=(y,j)$ and $(k,l)=(k,x)$. Without loss of generality, assume the first possibility. We claim that for any s, $1\leqslant s\leqslant y-1$, the vertex v_{sx} is either adjacent or nonadjacent to all of $v_{y1},v_{y2},\ldots,v_{y(x-1)}$ in G^c. Otherwise, there exist two vertices v_{yp} and v_{yq}, $1\leqslant p$, $q\leqslant x-1$ and $p\neq q$, such that v_{sx} is joined to v_{yp} and is not joined to v_{yq} in G^c. We color the vertices of G^c as follows: Color both v_{sx} and v_{yq} by c_{y+1} and color the remaining vertices v_{mn} by c_m. This coloring is an achromatic $(y+1)$-coloring for G^c, a contradiction.

Permute the rows of $T_1(xy-1;y,x)$, if necessary, so that every vertex in $\{v_{1x},v_{2x},\ldots,v_{rx}\}$ is adjacent in G^c to all the vertices in $Y=\{v_{y1},v_{y2},\ldots,v_{y(x-1)}\}$ and every vertex in $\{v_{(r+1)x},v_{(r+2)x},\ldots,v_{(y-1)x}\}$ is nonadjacent in G^c to all the vertices of Y. Clearly, such an r exists. Hence, $G\cong rK_x\cup H(x,y,r)$. The existence of nonadjacent vertices v_{ix} and v_{yl} in G^c implies that $y-1-r\geqslant 1$ i.e., $y\geqslant r+2$.

We now prove the converse. If $G\cong(y-1)K_x\cup K_{x-1}$, then $G^c\cong K_{y-1}(x)\vee K_{x-1}^c$, and therefore $\alpha(G^c)=y$. If $G\cong rK_x\cup H(x,y,r)$ with $y\geqslant r+2$, consider G^c in $T_1(xy-1;y,x)$. Clearly $\chi(G^c)=y$ and hence $\alpha(G^c)\geqslant y$. Suppose that $\alpha(G^c)>y$ and $\mathscr{C}$ is any optimal achromatic coloring of G^c. As the subgraph induced by $C_1=\{v_{i1}\colon 1\leqslant i\leqslant y\}$ in G^c is K_y, $\mathscr{C}(v_{i1})\neq\mathscr{C}(v_{j1})$ for $i\neq j$. Assume that $\mathscr{C}(v_{i1})=c_i$. Consider the vertex v_{ij} with $1\leqslant i\leqslant y-1$ and $2\leqslant j\leqslant x-1$. Since v_{ij} is adjacent to all the vertices in C_1 except v_{i1}, $\mathscr{C}(v_{ij})$ is either c_i or a new color, say, c_{y+1}. Suppose it is c_{y+1}, then as both the vertices v_{i1} and v_{ij} are adjacent to all the vertices of G^c except the vertices in $R_i=\{v_{i1},v_{i2},\ldots,v_{ix}\}$, the colors c_i and c_{y+1} occur only in $\mathscr{C}(R_i)$. But then there is no edge with color ends c_i and c_{y+1} in $\mathscr{C}$, a contradiction. Hence, $\mathscr{C}(v_{ij})=c_i$. Now consider $v_{y2},v_{y3},\ldots,v_{y(x-1)}$. These vertices receive either the color c_y or a new one. Suppose some vertex v_{yj}, $2\leqslant j\leqslant x-1$ receives c_{y+1}, then as $\mathscr{C}$ is achromatic, there must be an edge in G^c with color ends c_y and c_{y+1}. This implies that either c_y or c_{y+1} belongs to $\mathscr{C}(\{v_{1x}, v_{2x}, \ldots, v_{rx}\})$. But then there will be an edge with color ends either c_y and c_y or c_{y+1} and c_{y+1}, a contradiction. Hence, $\mathscr{C}(\{v_{y2}, \ldots, v_{y(x-1)}\}) = \{c_y\}$. Moreover, as the subgraph induced by $C_x = \{v_{ix}\colon 1 \leqslant i \leqslant y-1\}$ in G^c is K_{y-1}, any two distinct vertices of C_x cannot receive the color c_{y+1} and hence there exists exactly one vertex, say, v_{ix} in C_x with color c_{y+1}. But then, in G^c there is no edge with color ends c_i and c_{y+1}, a contradiction. Thus, $\alpha(G^c) = y$. □

Finally, we consider the case when $x = 3$.

Lemma 4. *Let $y \geqslant 2$ be any positive integer and let $G \in T_1(3y-1; 3, y)$. Then $\alpha(G^c) = y$ iff $G \cong (y-1)K_3 \cup K_2$, $(y-2)K_3 \cup (2K_2 \vee K_1)$ or $(y-2)K_3 \cup W_5$.*

Proof. Assume that $\alpha(G^c) = y$.

Claim 1. *For $i \neq y$, $j \neq y$ and $i \neq j$, $G^c[\{v_{i1}, v_{i2}, v_{i3}; v_{j1}, v_{j2}, v_{j3}\}] \cong K_{3,3}$.*

If this claim were not true, then, without loss of generality assume that $(v_{i1}, v_{j2}) \notin E(G^c)$. Color v_{i1} and v_{j2} by c_{y+1} and for the remaining vertices, color v_{mn} by c_m. This yields an achromatic $(y+1)$-coloring for G^c, a contradiction.

Claim 2. *For $i \neq y$, $(v_{y1}, v_{i3}) \in E(G^c)$ iff $(v_{y2}, v_{i3}) \in E(G^c)$.*

First let $(v_{y1}, v_{i3}) \in E(G^c)$. If $(v_{y2}, v_{i3}) \notin E(G^c)$, then color both v_{y2} and v_{i3} by c_{y+1} and for the remaining vertices, color v_{mn} by c_m. This yields an achromatic $(y+1)$-coloring for G^c, a contradiction. Hence $(v_{y2}, v_{i3}) \in E(G^c)$. The converse part follows by a similar argument.

Claim 3. *For $i \neq y$, $H' = G^c[\{v_{i1}, v_{i2}, v_{i3}; v_{y1}, v_{y2}\}] \cong K_{2,3}$, $C_4 \cup K_1$ or $2K_2 \cup K_1$.*

Suppose not, then, by Claim 2, H' is one of the following graphs of Fig. 2, in which the coloring is marked only for the vertices $v_{i1}, v_{i2}, v_{i3}, v_{y1}$ and v_{y2}. For the rest of the vertices v_{mn}, color them by c_m. This results in an achromatic $(y+1)$-coloring in each of the respective cases, a contradiction.

Claim 4. *For $i \neq y$, $j \neq y$ and $i \neq j$, at least one of $G_i = G^c[\{v_{i1}, v_{i2}, v_{i3}; v_{y1}, v_{y2}\}]$ and $G_j = G^c[\{v_{j1}, v_{j2}, v_{j3}; v_{y1}, v_{y2}\}]$ is isomorphic to $K_{2,3}$.*

Otherwise, by Claim 3, the following three cases arise. In each of these cases first color the vertex v_{mn} by c_m.

Case 4a: Both G_i and G_j are isomorphic to $C_4 \cup K_1$. Recolor v_{i3}, v_{j3} and v_{y2}, respectively, by c_y, c_{y+1} and c_{y+1}.

Case 4b: Both G_i and G_j are isomorphic to $2K_2 \cup K_1$. Recolor v_{i1}, v_{j1} and v_{y1}, respectively, by c_y, c_{y+1} and c_j.

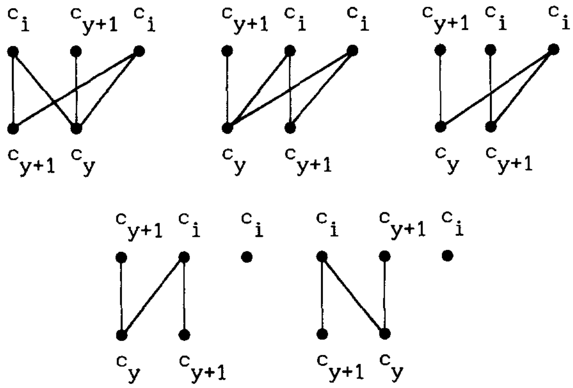

Fig. 2.

Case 4c: G_i is isomorphic to $C_4 \cup K_1$ and G_j is isomorphic to $2K_2 \cup K_1$. Recolor as in case 4a.

In each of the above cases, these recolorings together with Claim 1 result in an achromatic $(y+1)$-coloring for G^c, a contradiction.

The above claims ensure that $G \cong (y-1)K_3 \cup K_2$, $(y-2)K_3 \cup (2K_2 \vee K_1)$ or $(y-2)K_3 \cup W_5$.

We leave the converse as an exercise to the reader. □

3. Extremal graphs for the products

Theorem 5. *For a graph G on v vertices, $\chi(G)\alpha(G^c) = v$ iff G is either isomorphic to $dK_{v/d}$, where d is a positive divisor of v, or each component of G is isomorphic to a complete regular bipartite graph.*

Proof. If $\chi(G)\alpha(G^c) = v$, then $\chi(G)\chi(G^c) = v$ and therefore $\chi(G^c) = \alpha(G^c)$. Fink's result quoted in Section 1 implies that G is a graph in $T_1(v;\, v/d, d)$, where d is a positive divisor of v. As $G \in T_1(v; v/d, d)$, $G^c \in T_1(v; d, v/d)$. If $d=1$, then G is K_v and if $d = v$ then G is vK_1. So assume that $2 \leqslant d \leqslant v/2$. Suppose that $2 \leqslant d \leqslant v/3$, then by Lemma 1, $G \cong dK_{v/d}$. Suppose that $d = v/2$, then by Lemma 2, each component of G is isomorphic to a complete regular bipartite graph. The converse follows from Lemmas 1 and 2. □

Corollary 5.1. *For a graph G on v vertices, $\alpha(G)\alpha(G^c) = v$ iff G is isomorphic to either K_v, K_v^c, $2K_{v/2}$, or $K_{v/2,v/2}$.*

Proof. $\alpha(G)\alpha(G^c) = v$ implies that $\chi(G)\alpha(G^c) = v$ and $\alpha(G)\chi(G^c) = v$. By Theorem 5, both G and G^c are in $\{H$: H is either isomorphic to $dK_{v/d}$, where d is a positive

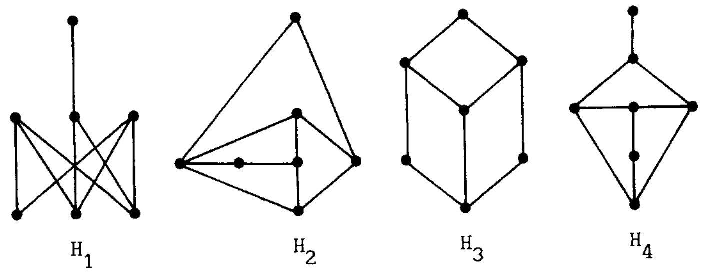

Fig. 3.

divisor of v, or each component of H is isomorphic to a complete regular bipartite graph}. Hence G is isomorphic to either $K_v, K_v^c, 2K_{v/2}$, or $K_{v/2,v/2}$. The converse holds trivially. □

Since $\alpha(K_{v/2,v/2})=2$ and $\psi(K_{v/2,v/2})=(v/2)+1$, we have

Corollary 5.2. *For a graph G on v vertices,* $\psi(G)\alpha(G^c)=v$ *iff G is isomorphic to either* K_v, K_v^c *or* $2K_{v/2}$.

Corollary 5.3. *For a graph G on v vertices,* $\psi(G)\psi(G^c)=v$ *iff G is isomorphic to either* K_v *or* K_v^c.

Corollary 5.4. *For a graph G on v vertices,* $\chi(G)\psi(G^c)=v$ *iff G is isomorphic to either* K_v, K_v^c *or* $K_{v/2,v/2}$.

Proof. By Theorem 5, G is either isomorphic to $dK_{v/d}$, where d is a positive divisor of v, or each component of G is isomorphic to a complete regular bipartite graph. Since for $2\leqslant d\leqslant(v/2)$, $\psi(K_d(v/d))>d=\alpha(K_d(v/d))$, we have $d=1$ or v. If $d=1$ or v then G is either K_v or K_v^c. If every component of G is isomorphic to a complete regular bipartite graph, then by hypothesis $\psi(G^c)=\alpha(G^c)=v/\chi(G)=v/2$. If $\omega(G)\geqslant 2$, then G^c contains two vertex disjoint copies of $K_{v/2}$. Now color the vertices of one copy $K_{v/2}$ of G^c by $c_1, c_2, \ldots, c_{v/2}$ and color all the vertices of the other copy $K_{v/2}$ of G^c by $c_{(v/2)+1}$. This shows that $\psi(G^c)\geqslant(v/2)+1$. Hence $\omega(G)=1$, i.e., $G\cong K_{v/2,v/2}$. The converse holds trivially. □

4. Extremal graphs for the sums

We begin with the case when $B=\psi$ in $A(G)+B(G^c)$.

Theorem 6. *For a graph G on v vertices,* $\chi(G)+\psi(G^c)=\lceil 2\sqrt{v}\rceil$ *iff G is one of the following graphs:* K_n *with* $n=1,2,3$; $K_{n,n}$ *with* $n=2,3,4,5$; $K_{n,n+1}$ *with* $n=2,3$; P_n

with $n=3,5$; nK_1 *with* $n=2,3$; $K(2,2,3)$; $K(1,3,3)$; $K_2\cup K_1$; $C_4\cup K_1$; $K_{3,3}\cup K_1$; $K_{2,3}-e$; $K_{3,4}-e$; W_5; $K_3^c\vee P_4$; $K_3^c\vee 2K_2$; H_i *with* $i=1,2,3,4$ *as in Fig.* 3.

Proof. Assume that $\chi(G)+\psi(G^c)=\lceil 2\sqrt{v}\rceil$. By Nordhaus–Gaddum inequality $\chi(G)+\chi(G^c)\geqslant\lceil 2\sqrt{v}\rceil$. As $\chi(G^c)\leqslant\psi(G^c)$, our hypothesis implies that $\chi(G)+\chi(G^c)=\lceil 2\sqrt{v}\rceil$ and $\chi(G^c)=\psi(G^c)$. Now by Fink's result (ii) quoted in the Introduction, G is a graph in $T_1(v;x,y)$ with $x+y=\lceil 2\sqrt{v}\rceil$. Further, $G\in T_1(v;x,y)$ implies that $\chi(G)=x$ and $\chi(G^c)=y$. We claim that $v+1<x+2y$. If not, $v+1\geqslant x+2y$. In this case, color the vertices of the first column of G^c in $T_1(v;\ y,x)$ with $c_1,c_2,\ldots,c_y$ in order, then color the rest of the vertices of the first row of $T_1(v;\ y,x)$ with c_{y+1} and the remaining vertices of G^c with colors $c_1,c_2,\ldots,c_y$ such that, for every color c_i, $1\leqslant i\leqslant y$, there exist at least two vertices in G^c with color c_i. (As $v-(x+y-1)\geqslant y$, such a coloring is possible.) This yields a pseudoachromatic $(y+1)$-coloring for G^c, a contradiction. Now $v+1<x+2y$ implies that $v+1\leqslant 4\sqrt{v}$ and hence $v\leqslant 13$. A search for all possible pairs (x,y) with $x+y=\lceil 2\sqrt{v}\rceil$, $v+1<x+2y$, $v\leqslant 13$ and $xy\geqslant v$ results in the collection of graphs that are listed in the statement. The converse can be easily verified. □

As a consequence of the above theorem, for any graph G with at least 11 vertices, $\psi(G)+\psi(G^c)\geqslant\alpha(G)+\psi(G^c)\geqslant\chi(G)+\psi(G^c)\geqslant\lceil 2\sqrt{v}\rceil+1$.

We now consider the case when $B(G^c)=\alpha(G^c)$.

Theorem 7. *For a graph* G *on* m^2 *vertices,* $\chi(G)+\alpha(G^c)=2m$ $(=\lceil 2\sqrt{v}\rceil)$ *iff* G *is isomorphic to* mK_m *or* C_4.

Proof. If $\chi(G)+\alpha(G^c)=2m$, then by Fink's result (ii), $G\in T_1(m^2;x,y)$ with $x+y=2m$. As $xy\geqslant m^2$, $(x,y)=(m,m)$. Also $\alpha(G^c)=\chi(G^c)=y=m$. If $m\geqslant 3$, then by Lemma 1, $G\cong mK_m$. If $m=2$, then by Lemma 2, G is either $2K_2$ or C_4. If $m=1$, then G is K_1. The converse holds trivially. □

Theorem 8. *For a graph* G *on* m^2+m-1 *vertices,* $\chi(G)+\alpha(G^c)=2m+1$ *iff* G *is one of the following graphs*: $mK_m\cup K_{m-1}$, $m\geqslant 4$; $(m-1)K_{m+1}\cup K_m$, $m\geqslant 3$; $rK_m\cup H(m,m+1,r)$, $m\geqslant r+1$ *and* $m\geqslant 4$; $rK_{m+1}\cup H(m+1,m,r)$, $m\geqslant r+2$ *and* $m\geqslant 3$; $3K_3\cup K_2$; $2K_3\cup(2K_2\vee K_1)$; $2K_3\cup W_5$; $2K_2\cup K_1$; $P_3\cup K_2$; $C_4\cup K_1$; P_5; $K_{2,3}-e$; $K_{2,3}$; $K_3\cup K_2$; $(2K_2\vee K_1)$ *and* W_5.

Proof. If $\chi(G)+\alpha(G^c)=2m+1$, then $G\in T_1$ $(m^2+m-1;x,y)$ with $x+y=2m+1$. Since $xy\geqslant m^2+m-1$, solving for x and y results in (x,y) to be either $(m,m+1)$ or $(m+1,m)$.

Case 1: $(x,y)=(m,m+1)$

We have $\alpha(G^c)=\chi(G^c)=y=m+1$. If $m\geqslant 4$, then by Lemma 3, $G\cong mK_m\cup K_{m-1}$ or $rK_m\cup H(m,m+1,r)$, with $m\geqslant r+1$. If $m=3$, then by Lemma 4, $G\cong 3K_3\cup K_2$, $2K_3\cup(2K_2\vee K_1)$ or $2K_3\cup W_5$. If $m=2$, then $G\cong 2K_2\cup K_1$; $P_3\cup K_2$; $C_4\cup K_1$; P_5; $K_{2,3}-e$ or $K_{2,3}$. Note that $m\neq 1$.

Case 2: $(x,y)=(m+1,m)$

If $m\geqslant 3$, then by Lemma 3, $G\cong(m-1)K_{m+1}\cup K_m$ or $rK_{m+1}\cup H(m+1,m,r)$, $m\geqslant r+2$. If $m=2$ then by Lemma 4, $G\cong K_3\cup K_2$; $(2K_2\vee K_1)$ or W_5.

It is easy to verify the converse. □

Theorem 9. *For a graph G on m^2+m vertices, $\chi(G)+\alpha(G^c)=2m+1$ iff G is one of the following graphs*: mK_{m+1} *with* $m\geqslant 2$; $(m+1)K_m$ *with* $m\geqslant 3$; $K_{3,3}$; $C_4\cup K_2$; $3K_2$; K_2 *and* K_2^c.

Proof. If $\chi(G)+\alpha(G^c)=2m+1$, then $G\in T_1$ $(m^2+m;x,y)$ with $x+y=2m+1$. Since $xy\geqslant m^2+m$, solving for x and y we get (x,y) to be either $(m+1,m)$ or $(m,m+1)$.

Case 1: $(x,y)=(m+1,m)$

If $m\geqslant 2$, then by Lemma 1, $G\cong mK_{m+1}$. If $m=1$, then G is K_2.

Case 2: $(x,y)=(m,m+1)$

If $m\geqslant 3$, then by Lemma 1, $G\cong(m+1)K_m$. If $m=2$, then by Lemma 2, $G\cong K_{3,3}$, $C_4\cup K_2$ or $3K_2$. If $m=1$, then G is K_2^c.

Once again, it is easy to check the converse. □

Theorem 10. *For a graph G on m^2+2m vertices, $\chi(G)+\alpha(G^c)=2m+2$ iff G is one of the following graphs*: mK_{m+2}; $mK_{m+1}\cup K_m$ *with* $m\geqslant 3$; $rK_{m+1}\cup H(m+1,m+1,r)$ *with* $m\geqslant 3$ *and* $m\geqslant r+1$; $(m+2)K_m$ *with* $m\geqslant 3$; $2K_3\cup K_2$; $K_3\cup(2K_2\vee K_1)$; $K_3\cup W_5$; $K_2\cup K_1$; P_3; $4K_2$; $2K_2\cup C_4$; $K_2\cup K_{3,3}$; $2C_4$; $K_{4,4}$ *and* K_3^c.

Proof. If $\chi(G)+\alpha(G^c)=2m+2$, then $G\in T_1$ $(m^2+2m;x,y)$ with $x+y=2m+2$. Since $xy\geqslant m^2+2m$, we see that (x,y) to be either $(m+2,m)$, $(m+1,m+1)$ or $(m,m+2)$.

Case 1: $(x,y)=(m+2,m)$

If $m\geqslant 2$, then by Lemma 1, $G\cong mK_{m+2}$. If $m=1$, then $G\cong K_3$.

Case 2: $(x,y)=(m+1,m+1)$

If $m\geqslant 3$, then by Lemma 3, $G\cong mK_{m+1}\cup K_m$ or $rK_{m+1}\cup H(m+1,m+1,r)$ with $m\geqslant r+1$. If $m=2$, then by Lemma 4, $G\cong 2K_3\cup K_2$, $K_3\cup(2K_2\vee K_1)$ or $K_3\cup W_5$. If $m=1$, then $G\cong K_2\cup K_1$ or P_3.

Case 3: $(x,y)=(m,m+2)$

If $m\geqslant 3$, then by Lemma 1, $G\cong(m+2)K_m$. If $m=2$, then by Lemma 2, $G\cong 4K_2$; $2K_2\cup C_4$; $K_2\cup K_{3,3}$; $2C_4$; $K_{4,4}$. If $m=1$, then G is K_3^c. □

From Theorems 7–10, one can deduce that for any graph G on v vertices, $v\in\{m^2,m^2+m-1,m^2+m,m^2+2m\}$, $m\geqslant 3$, $\alpha(G)+\alpha(G^c)\geqslant\lceil 2\sqrt{v}\rceil+1$.

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